Inductive approach to the representation theory of symmetric groups

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1 Introduction

This topic was suggested to me by Vera Serganova, as part of a graduate representation theory course at UC Berkeley.

The project is based on the Vershik-Okounkov approach to the representation theory of symmetric groups. It first appeared in Russian in their paper back in 1996, which was later modified and translated into English in 2005 [5]. We base our exposition on the latter paper. The approach that Vershik and Okounkov proposed corrects many drawbacks of the conventional approach to the representation theory of symmetric groups. In particular, it takes into advantage the properties of the groups and uses the fact that they form an inductive chain (with the natural embedding $S_{n-1} \hookrightarrow S_n$) to build the theory inductively. This way, Young diagrams and the branching rule appear naturally in such theory, making the big picture clear and complete. To construct this theory, we will introduce the notions of Gelfand-Tsetlin algebra, Gelfand-Tsetlin basis, and Young-Jucys-Murphy elements. The appearance of these concepts is due to the fact that symmetric groups are Coxeter groups, and taking into an account their intrinsic structure is the foundation of this approach.

2 Branching graph and Gelfand-Tsetlin algebra

In this section we introduce the first building blocks in the representation theory of symmetric groups: the branching graph, Gelfand-Tsetlin Algebra, and Gelfand-Tsetlin basis. Given an inductive chain of finite groups

$$\{0\} = G(0) \subseteq G(1) \subseteq G(2) \subseteq \cdots$$

let $G(i)^\wedge$ denote the set of all distinct (up to isomorphism) irreducible representations of the group $G(i)$ in the chain. For any group G and any irreducible representation ρ of G, denote by V^ρ the the corresponding G-module. We define the *branching graph* $\Gamma = (V, E)$ of the chain as follows. Let $V = \{\rho : \rho \in G(i)^\wedge, i \in \mathbb{N}\}$, i.e. the vertices of Γ correspond to irreducible representations of the groups in the chain. Let $\mu \in G(i-1)^\wedge$ and $\lambda \in G(i)^\wedge$ be irreducible representations. In the branching graph, we join μ and λ by k directed edges $\mu \to \lambda$, where

$$k = \dim \operatorname{Hom}_{G(i-1)}(V^\mu, \operatorname{Res} V^\lambda) = (\mu, \operatorname{Res} \lambda)_{G(i-1)},$$

and no other vertices are joined by edges. Note that by construction, if $i \leq k$, the multiplicity of $\mu \in G(i)^\wedge$ in $\operatorname{Res}_{G(i)} \lambda$ for some $\lambda \in G(k)^\wedge$ is nonzero iff there is a directed path from μ to λ in the branching graph.

From the definition above, it is clear that the branching graph is in fact a directed multigraph. We say that *branching is simple* (or *multiplicities are simple*) for some inductive family of groups if the branching graph is simple, i.e. when it has no multiple edges. If the branching is simple for the inductive family $\{G(i)\}_{i\in\mathbb{N}}$, this means that at every stage of the graph, the multiplicity of an irreducible representation μ of $G(i-1)$ in $\operatorname{Res}_{G(i-1)} \lambda$ where $\lambda \in G(i)^\wedge$ is either 0 or 1. Hence the usual decomposition

$$\operatorname{Res}_{G(i-1)} V^\lambda = \bigoplus_{\substack{\mu\to\lambda \\ \mu\in G(i-1)^\wedge}} V^\mu$$

is canonical. Iterating this decomposition i times, we obtain the canonical decomposition

$$\operatorname{Res}_{G(0)} V^\lambda = \bigoplus_T V_T$$

where each V_T is a $G(0)$-module and the sum is taken over all possible paths

$$T = (\varnothing = \lambda^{(0)} \to \lambda^{(1)} \to \cdots \to \lambda^{(i)} = \lambda)$$

of the branching graph starting at $\varnothing$ ending in λ. Note that each V_T is a one-dimensional vector space, so we may pick a unit vector v_T (with respect to the $G(i)$-invariant inner product) in each V_T and obtain a basis of V^λ. This basis is called the *Gelfand-Tsetlin basis (GZ basis)*. It is named after the mathematicians Israel Gelfand and Michael Tsetlin. Israel Gelfand was a prominent Jewish Soviet mathematician who made significant contributions to group theory, representation theory and functional analysis. He was a recipient of many awards, including the Order of Lenin and the first Wolf Prize. Gelfand was described to be "among the greatest mathematicians of the 20th century" by *The New York Times*, having made his scientific impact both through his work and numerous students. Michael Tsetlin, also spelled Zetlin (hence the abbreviation GZ) was also a Russian mathematician and physicist, whose research was in cybernetics. Gelfand-Tsetlin bases are very important and widely used in theoretical physics [6],[7].

For each $i \in \mathbb{N}$, denote by $\mathbb{C}[G(i)]$ the group algebra of $G(i)$, and let $Z(i)$ be the center of this algebra. Define the *Gelfand-Tsetlin algebra (GZ algebra)* $\mathrm{GZ}(i)$ of the inductive family of groups $\{G(i)\}_{i\in\mathbb{N}}$ to be the subalgebra of $\mathbb{C}[G(i)]$ generated by $Z(1), \cdots, Z(i)$. In symbols,

$$\mathrm{GZ}(i) = \langle Z(1), \cdots, Z(i)\rangle.$$

It is clear from this definition that $\mathrm{GZ}(i)$ is commutative. Moreover, it turns out that the algebra $GZ(i)$ is a maximal commutative subalgebra of the group algebra $\mathbb{C}[G(i)]$. Furthermore, it coincides with the subalgebra of functions on $G(i)$ whose Fourier transforms are diagonalized by certain Gelfand–Tsetlin bases. Further, it can be shown that every element in the Gelfand–Tsetlin basis of V^λ is a common eigenvector for all operators $\lambda(f)$ with $f \in \mathrm{GZ}(i)$. In particular, it is uniquely determined, up to a scalar factor, by the corresponding eigenvalues. For more details and proofs of these facts, refer to Section 2.2 of [1].

3 Branching for symmetric groups is simple

In this section we prove a well-known result that the branching graph for symmetric groups is simple, using the theory of centralizers and involutive algebras. While the concepts in previous section were defined for general inductive chains of finite groups, we are interested in the particular case when $G(n) = S_n$ for all $n \in \mathbb{N}$. Hence, from now on, assume $G(n) = S_n$, the symmetric group on n letters.

For any semi-simple finite-dimensional $\mathbb{C}$-algebra M and its semi-simple subalgebra N, define the *centralizer* $Z(M, N)$ of the pair (M, N) to be

$$Z(M, N) = \{m \in M : mn = nm \text{ for all } n \in N\}.$$

Note that in the original paper [5], N is not necessarily assumed to be semi-simple, so the assumption here is weaker, but sufficient for our purposes.

Theorem 1 (Artin-Wedderburn theorem)**.** Any semi-simple $\mathbb{C}$-algebra decomposes as a direct sum of matrix algebras over $\mathbb{C}$.

Theorem 2 (Double Centralizer theorem)**.** Let A be a finite-dimensional simple algebra with the property $Z(A) = \mathbb{C}$ and let $B \subseteq A$ be a simple subalgebra. Then $C := Z(A, B)$ is simple, $Z(A, C) = B$, and $\dim_{\mathbb{C}}(A) = \dim_{\mathbb{C}}(B) \cdot \dim_{\mathbb{C}}(C)$.

Theorem 1 is a classical theorem in algebra and the proof of Theorem 2 can be found in [2]. The following criterion will be very useful.

Proposition 1 ([5], Section 1)**.** The centralizer $Z(M, N)$ is semi-simple and the following conditions are equivalent:

(1) The restriction of any finite-dimensional irreducible complex representation of M to N has simple multiplicities.

(2) The centralizer $Z(M, N)$ is commutative.

Proof. We present a proof different from the proof in the original paper [5]. The following proof is due to Wigner (outlined in [4]) and it employs some nice results from algebra. First, note that by assumption, M is a semi-simple complex algebra, so by Theorem 1, it decomposes as a direct sum of matrix algebras over $\mathbb{C}$. Thus, we may write

$$M = \bigoplus_{i=1}^{k} M_i$$

where each M_i is a complex matrix algebra. Thus, we may identify the elements of M with tuples $(m_1, \cdots, m_k)$ with each $m_i \in M_i$. For each $i \in [k]$, define the projection map $\pi_i : M \to M_i$ from M onto the ith component. Since N is assumed to be a semi-simple subalgebra of M and since π_i is clearly a homomorphism, the image $\pi_i(N) = N_i$ is also a semi-simple algebra.

Next, it is easily seen that $Z(M, N) = \bigoplus_{i=1}^{k} Z(M_i, N_i)$. Since each M_i is a matrix algebra over a field, it it simple. Moreover, if M_i is an algebra of 1×1 matrices, then $M_i \cong \mathbb{C}$ and $Z(M_i) = \mathbb{C}$; if M_i is a set of $r \times r$ matrices with $r > 1$, then the only elements in M_i commuting with everything else are $c \cdot I_r$ where $c \in \mathbb{C}$, in which case again $Z(M_i) = \mathbb{C}$. Thus, M_i is a simple central algebra and N_i is its semi-simple subalgebra, so by Theorem 2, we conclude that $Z(M_i, N_i)$ is simple for each i. It then follows (by definition of semisimple) that $Z(M, N)$ is semi-simple, as desired.

We now show that (1) is equivalent to (2). For all $i \in [k]$, denote by V_i the set of all tuples $(m_1, \cdots, m_k)$ where $m_j = 0$ for all $j \neq i$ and all entries of m_i, except for the entries in the first column, are zero. It is straightforward to verify (in fact, we had a similar problem on a homework) that $V_1, \cdots, V_k$ are all the distinct irreducible M-modules. Moreover, since each V_i embeds into N_i, the decomposition of V_i into irreducible N-modules is the same as its decomposition into irreducible N_i-modules. Now, from the decomposition of M, we have that $Z(M, N)$ is commutative iff $Z(M_i, N_i)$ is commutative for all $i \in [k]$. Since the number of irreducible representations is the number of conjugacy classes, the latter holds iff all irreducible representations of $Z(M_i, N_i)$ are one-dimensional for all i. Hence, we need to show that all irreducible representations of $Z(M_i, N_i)$ have dimension 1 iff the restriction of any finite-dimensional irreducible complex representation of M_i to N_i has simple multiplicities.

For the forward direction, assume all irreducible representations of $Z(M_i, N_i)$ are one-dimensional. Let U and V be irreducible representations of M_i and N_i, respectively. Then $\mathrm{Hom}_{N_i}(U, V)$ is an irreducible representation of $Z(M_i, N_i)$, so it must have dimension 1. But $\dim \mathrm{Hom}_{N_i}(U, V)$ is precisely the multiplicity of V in $\mathrm{Res}_{N_i} U$, so multiplicity is simple. For the other direction, suppose $Z(M_i, N_i)$ has an irreducible representation of dimension more than 1. Again, let U and V be irreducible representations of M_i and N_i. Since by Schur's lemma we have $\mathrm{End}_{M_i}(U) \cong M_i$, this implies $\mathrm{End}_{N_i}(U) \cong Z(M_i, N_i)$. But then the decomposition of $\mathrm{Res}_{N_i} U$ into simple N_i-modules $\{W_j\}$ must have some simple module appearing more than once. But since $Z(M_i, N_i)$ is a direct sum of modules of the form $\mathrm{Hom}_{N_i}(W_r, W_s)$, the multiplicity of N_i in $\mathrm{Res}_{N_i}(U)$ is more than 1, implying branching is not simple. Hence, for each i the conclusion holds, and by the discussion above, the conclusion holds for M and N. $\square$

Define $Z(\ell, k) := Z(\mathbb{C}[S_{\ell+k}], \mathbb{C}[S_\ell])$. We will show that $Z(n-1, 1) = Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}])$ is commutative, then use Proposition 1 to conclude that at each step of the branching graph the restriction is simple, and hence the branching for symmetric groups is simple. Before we can do this, we need several lemmas.

Lemma 1 ([5], Section 2)**.** Every element $g \in S_n$ is conjugate to its inverse, that is $g^{-1} = hgh^{-1}$ for some $h \in G$. Moreover, such h may be chosen from the subgroup S_{n-1}.

Proof. Let $g \in S_n$. Writing g as a product of disjoint cycles $g = c_1 \cdots c_k$, it is clear that $g^{-1} = c_k^{-1} \cdots c_1^{-1}$ and $hgh^{-1} = (hc_1h^{-1}) \cdots (hc_kh^{-1})$. Recall that in the symmetric group, two elements are conjugate iff they have the same cycle type. Since for each cycle $c_i = (j_1\ j_2 \cdots j_r)$ in g, we have $c_i^{-1} = c_i^{r-1} = (j_r\ j_{r-1} \cdots j_1)$, we get that g and g^{-1} have the same cycle type, and so there exists some $h \in S_n$ with $hgh^{-1} = g^{-1}$. To prove the second assertion, let $g' \in S_{n-1}$ be the permutation on $\{1, \cdots, n-1\}$ induced from g by removing n from the cycle containing it. Then by the previous claim, there is some $h \in S_{n-1}$ such that $(g')^{-1} = hg'h^{-1}$. Note also that for every cycle $c_i = (j_1\ j_2 \cdots j_r)$ of g', $hc_ih^{-1} = (h(j_1)\ h(j_2) \cdots h(j_r))$. Hence, regarding h a permutation in S_n (by treating n as a fixed point), we obtain $hgh^{-1} = g^{-1}$, as desired. $\square$

We say that an algebra A over $\mathbb{C}$ is *involutive* if it has a bijective map to itself $x \mapsto x^*$ such that for all $a \in \mathbb{C}$, $x, y \in A$, we have: $(x+y)^* = x^* + y^*$, $(ax)^* = \overline{a}x^*$, $(xy)^* = y^*x^*$, and $(x^*)^* = x$. We say an element $x \in A$ is *self-adjoint* if $x^* = x$.

Lemma 2. Every element x in an involutive algebra can be written as $x = u + iv$ where u and v are self-adjoint.

Proof. Re-write
$$x = \frac{x + x^*}{2} + i\frac{-i(x - x^*)}{2}.$$
Since $(x+x^*) = x^*+x$ and $(-i(x-x^*))^* = i(x^*-x) = -i(x-x^*)$, both $(x+x^*)/2$ and $-i(x-x^*)/2$ are self-adjoint, and conclusion follows. □

Lemma 3 ([5], Section 2)**.** An involutive algebra over $\mathbb{C}$ is commutative iff all of its element commute with its adjoint. If any real element of algebra is self-conjugate then it is commutative.

Proof. The forward direction is trivial. Let A be an involutive algebra over $\mathbb{C}$ in which every element commutes with its adjoint. Let $u, v \in A$ such that $u = u^*$ and $v = v^*$. Then $(u + iv)^* = u^* - iv^* = u - iv$. Since $u + iv$ is an element of the algebra, it commutes with its adjoint, so we get $(u+iv)(u-iv) = (u-iv)(u+iv)$. Expanding, we obtain that $uv = vu$. Hence, any two self-adjoints commute. Now let $x, y \in A$ be arbitrary. By Lemma 2, we may write $x = u + iv$ and $y = a + ib$ where u, v, a, b are self-adjoint. We have

$$xy = (u + iv)(a + ib) = ua + i(va) + i(ub) - vb = au + i(av) + i(bu) - bv = (a + ib)(u + iv) = yx$$

and hence A is commutative, as desired.

To prove the second claim, let A be an involutive algebra over $\mathbb{C}$ that is a complex hull of a real algebra. Suppose every real element of A is self-conjugate. By the previous claim, it suffices to show that every element $x \in A$ commutes with its adjoint. It is clear that we may write $x = u+iv$ for some real elements u, v. Since u and v are real, they are self-adjoint by assumption, so $(u+iv)^* = u-iv$. But then uv is also real and thus self-adjoint, so $uv = (uv)^* = v^*u^* = vu$. Therefore,

$$xx^* = (u + iv)(u - iv) = u^2 - v^2 = (u - iv)(u + iv) = x^*x$$

and the conclusion follows. □

Theorem 3 ([5], Section 2)**.** The branching of the chain $\mathbb{C}[S_1] \subseteq \cdots \subseteq \mathbb{C}[S_n]$ is simple.

Proof. We show that every real element in $Z(n-1, 1) \subseteq \mathbb{C}[S_n]$ is self-conjugate. Let $f = \sum_i r_i g_i \in \mathbb{R}[S_n]$ be an arbitrary real element. By definition of centralizer, f commutes with every element of $\mathbb{C}[S_n]$, so in particular with all the elements in S_n. Since S_n is a basis of the group algebra, the above coefficients r_i in the expansion of f are unique, so the automorphism $\varphi_h : f \mapsto hfh^{-1}$ has f as a fixed point for every $h \in S_{n-1}$. For each g_i in the expansion of f, let $h_i \in S_{n-1}$ such that $g_i^{-1} = h_i g_i h_i^{-1}$ as in Lemma 1. Then since $\varphi_i : f \mapsto h_i f h_i^{-1}$ fixes f, the expansion of f must (by uniqueness) contain the terms $c_i g_i$ and $c_i g_i^{-1}$ for every i. But then f is a fixed point of the anti-automorphism $x \mapsto x^{-1}$, so $f^* = f$ as desired. By Lemma 3, $Z(n-1, 1)$ is commutative. Applying Proposition 1 inductively n times, we conclude that the branching for symmetric groups is simple. □

4 Young-Jucys-Murphy elements

In the previous section we proved that the centralizer $Z(n-1,1)$ is commutative. In this section, we give a much more detailed description of its structure and properties, by introducing Young-Jucys-Murphy elements. Set $X_1 = 0$. For each $i = 2, \cdots, n$, we define

$$X_i := (1\ i) + (2\ i) + \cdots + (i-1\ i) \in \mathbb{C}[S_i],$$

which are called *Young-Jucys-Murphy elements (YJM elements).* Note that X_i is the sum of all transpositions that appear in S_i, but not in S_{i-1}, so X_i is the sum of all transpositions in S_i minus the sum of all transpositions in S_{i-1}. Moreover, it is easy to see that the sum of all transpositions in a symmetric group commutes with every element of the group algebra, so $X_i \in Z(i) - Z(i-1) \subseteq GZ(i)$. Let $\sigma \in S_n$, and let P_σ denote the sum of all permutations in S_n that have the same cycle structure as σ. Let P_i denote the sum of all cycles of length i in S_n. We will need the following lemma:

Lemma 4 (Inspired by [4], Section 2)**.** The center $Z(n)$ of the group algebra $\mathbb{C}[S_n]$ is generated by the elements P_σ where σ is a cycle in S_n.

Proof. First, we show that the set $\{P_\sigma : \sigma \in S_n\}$ is a linear basis of $Z(n)$. Note that for any $\sigma, \tau \in S_n$, we have $\tau P_\sigma \tau^{-1} = P_\sigma$, since conjugation doesn't change cycle type and thus simply permutes the summands in P_σ. Thus, $P_\sigma \in Z(n)$ for each $\sigma \in S_n$, and the above claim makes sense. Next, note that it is impossible to add elements of different cycle types with nonzero coefficients and get zero, so P_σ's are linearly independent. It remains to show they span $Z(n)$. Let $f = \sum_{\sigma \in S_n} a_\sigma \sigma \in Z(n)$ (note this decomposition is unique), and let $\tau \in S_n$. Since f is in the center of the group algebra $\mathbb{C}[S_n]$, we have $f\tau = \tau f$, which means $\tau f \tau^{-1} = f$, i.e. f is invariant under conjugation by the elements in S_n. Hence, in the expansion of f, conjugate elements have the same coefficients (that is $a_\sigma = a_{\tau\sigma\tau^{-1}}$). Since elements of S_n are conjugate iff they have the same cycle type, we have that $a_\rho = a_\mu$ whenever ρ and μ have the same cycle type (or correspond to the same *partition*). Then, factoring out the coefficients in f, we obtain $f = \sum_{\sigma \in P(n)} a_\sigma P_\sigma$, where the sum is taken over all partitions of n, denoted by $P(n)$. Hence, indeed $\{P_\sigma : \sigma \in S_n\}$ is a linear basis.

Next, note that $\langle P_1, \cdots, P_n \rangle \subseteq Z(n)$ by a previous remark, so it remains to show the other containment. Since $\{P_\sigma : \sigma \in S_n\}$ is a linear basis of $Z(n)$, it suffices to show that $P_\sigma \in \langle P_1, \cdots, P_n \rangle$ for all $\sigma \in S_n$. We proceed by induction on the number of elements σ moves, k. Suppose $k = 0$ or $k = 1$. Then σ is the identity permutation, and clearly $\mathrm{id} \in \langle P_1, \cdots, P_n \rangle$, so the base case holds. For the inductive case, let σ be a permutation in S_n that moves k elements, and suppose the claim holds for any permutations that move at most $k-1$ elements. Write $\sigma = \sigma_1 \cdots \sigma_r$ as a product of disjoint cycles σ_i, each of length n_i. Consider the product $P_{\sigma_1} \cdots P_{\sigma_r} = P_{n_1} \cdots P_{n_r} \in Z(n)$. Let ρ be a summand in $P_{n_1} \cdots P_{n_r}$. Note that if ρ is made up of disjoint cycles, it produces a summand in P_σ. If ρ is made up of non-disjoint cycles, it is a summand in some P_τ where $\tau \in S_n$ moves strictly less elements than σ. Using this analysis, we may write

$$P_{n_1} \cdots P_{n_r} = P_{\sigma_1} \cdots P_{\sigma_r} = a_\sigma P_\sigma + \sum_\tau a_\tau P_\tau,$$

where the last sum is taken over all $\tau \in S_n$ that move strictly less elements than σ. By the inductive hypothesis, the last sum is in $\langle P_1, \cdots, P_n \rangle$. Then

$$P_\sigma = \frac{1}{a_\sigma}\left(P_{n_1} \cdots P_{n_r} - \sum_\tau a_\tau P_\tau\right) \in \langle P_1, \cdots, P_n \rangle,$$

and the conclusion follows. $\square$

The above lemma could also be deduced from the fact that the power sums generate the algebra of symmetric functions. For more details, refer to the Chapter 1 of [3].

Theorem 4 ([5], Section 2)**.** The center of the group algebra $\mathbb{C}[S_n]$ is contained in the algebra generated by the center of $\mathbb{C}[S_n]$ and the nth YJC element, i.e.

$$Z(n) \subseteq \langle Z(n-1), X_n \rangle.$$

Proof. For $k, r \in \mathbb{N}^+$, denote by $P_k(r)$ the sum of all k-cycles in S_r. By Lemma 4, it suffices to show that for every $k \in [n]$, $P_k(n) \in \langle Z(n-1), X_n \rangle$. We proceed by induction on k. Note that X_n is the difference of the sums of all transposition in S_n and S_{n-1}, so we write:

$$X_n = \sum_{i=1}^{n-1} (i\ n) = \sum_{\substack{i \neq j \\ i,j=1}}^{n} (i\ j) - \sum_{\substack{i \neq j \\ i,j=1}}^{n-1} (i\ j).$$

It is clear that the first sum is an element of $Z(n)$ and the second sum is an element of $Z(n-1)$. Then we get $P_2(n) \in X_n + Z(n-1) \subseteq \langle Z_{n-1}, X_n \rangle$, so the claim holds for $k = 2$. In a similar manner, it is easy to show that the claim holds for $k = 3$, by writing out the expression of X_n^2. Suppose, the claim holds for $k - 1$. Observe that we may write:

$$X_n \cdot \sum_{i_1, \cdots, i_k = 1}^{n} (i_1\ \cdots\ i_{k-1}\ n) = \sum_{\substack{i \neq i_j \\ j=1,\cdots n-1}} (i\ n)(i_1\ \cdots i_{k-1}\ n) + \sum_{i, i_1, \cdots, i_{k-1}} (i\ i_1 \cdots\ i_{k-1}\ n).$$

Denote the expression on the left-hand side by A, the first sum on the right-hand side by B and the second sum by C. Let

$$D := \sum_{i,j,i_1,\cdots,i_{k-1}}^{n-1} (i\ j)(i_1\ \cdots\ i_{k-1}) \in Z(n-1).$$

By the inductive hypothesis, $B + D \in \langle Z(n-1), X_n \rangle$, so $C = A - (B + D) - D \in \langle Z(n-1), X_n \rangle$ as well. But then we get

$$P_k(n) = \sum_{i, i_1, \cdots, i_{k-1}} (i\ i_1 \cdots\ i_{k-1}\ n) + \sum_{i, i_1, \cdots, i_k} (i\ i_1 \cdots\ i_{k-1}\ i_k) \in \langle Z(n-1), X_n \rangle$$

and the claim follows. $\square$

Corollary 1 ([5], Section 2)**.** The GZ algebra is generated by YJM elements, i.e.

$$\mathrm{GZ}(n) = \langle X_1, X_2, \cdots, X_n \rangle.$$

Proof. We proceed by induction on n. When $n = 2$, we have $\mathrm{GZ}(2) = \mathbb{C}[S_1] = \mathbb{C}[1, (1\ 2)] = \langle (1\ 2) \rangle = \langle X_1, X_2 \rangle$, so the base cas holds. Suppose that the claim holds for $n - 1$, i.e. $\mathrm{GZ}(n-1) = \langle X_1, \cdots, X_{n-1} \rangle$. Hence, it suffices to show that $\mathrm{GZ}(n) = \langle \mathrm{GZ}(n-1), X_n \rangle$. We prove it by double containment. By definition of $GZ(n)$, it is clear that $Z(i) \subseteq \mathrm{GZ}(n-1)$ or all $i \in [n-1]$, hence for the forward inclusion it suffices to show $Z(n) \subseteq \langle \mathrm{GZ}(n-1), X_n\rangle$. By Theorem 4, we get that $Z(n) \subseteq \langle Z(n-1), X_n\rangle \subseteq \langle \mathrm{GZ}(n-1), X_n\rangle$, and so the forward containment follows. The reverse containment follows from the facts that $\mathrm{GZ}(n-1) \subseteq \mathrm{GZ}(n)$ by definition and that $X_n \in Z(n) - Z(n-1) \subseteq \mathrm{GZ}(n)$. $\square$

Theorem 5 ([5], Section 2)**.** The centralizer $Z(n-1,1)$ is generated by $Z(n-1)$ and X_n, i.e.

$$Z(n-1,1) = \langle Z(n-1), X_n\rangle.$$

Proof. We proceed by double containment. For the forward containment, note that a basis in $Z(n-1,1)$ can be written as a union of a basis of the elements in the center of $\mathbb{C}[S_n]$ and the remaining elements in S_n that commute with everything in $\mathbb{C}[S_{n-1}]$. Elements of the latter type must contain n in one of the cycles, and hence are generated by elements of the form

$$\sum_{\substack{i_j^{(\ell)}=1 \\ \text{all distinct}}}^{n-1} (i_1^{(1)} \cdots i_{k_1}^{(1)}\ n)(i_1^{(2)} \cdots i_{k_2}^{(2)}) \cdots (i_1^{(r)} \cdots i_{k_r}^{(r)}). \tag{1}$$

Note that summing each of the expressions above with the sum of all permutations of the same cycle type that don't contain n (note each such sum is in $Z(n-1)$), we obtain every P_σ for $\sigma \in S_n$. Since we know that each $P_\sigma \in Z(n)$, we get that each of the expressions in (1) are contained in $Z(n) - Z(n-1)$. Hence, $Z(n-1,1) \subseteq \langle Z(n-1), Z(n)\rangle$. But since by Theorem 4, $Z(n) \subseteq \langle Z(n-1), X_n\rangle$, we have $Z(n-1,1) \subseteq \langle Z(n-1), X_n\rangle$, as desired. For the reverse containment, first note that $Z(n-1) \subseteq Z(n-1,1)$ simply by definition. Moreover, for any cycle $s \in S_{n-1}$, we have $(s(i)\ n)s = s(i\ n)$ for all i appearing in s and $(i\ n)s = s(i\ n)$ for all i not appearing in s. Therefore, all elements of X_n commute with $\mathbb{C}[S_{n-1}]$, and the conclusion follows. $\square$

5 Main result: relationship between symmetric groups and Young diagrams

Note that Theorem 5 in the previous section provides another way of seeing that the centralizer $Z(n-1,1)$ is commutative. Now, let $\rho \in S_n^\wedge$ be any irreducible representation. Let $\{v_T\}$ be the Gelfand-Tsetlin basis of V^ρ with respect to the chain $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n$. We call this basis the *Young basis* for V^ρ.

We have already mentioned that the Gelfand-Tsetlin algebra is a maximal commutative subalgebra of $\mathbb{C}[S_n]$. Combining this fact with Corollary 1 (see [1] for details), one may check that every element of the Young basis is a common eigenvector of $\rho(X_i)$ for $i \in [n]$, determined up to a scalar by the eigenvalues corresponding to these elements. This fact suggests the following definition. For any vector v_T in the Young basis of V^ρ, define the *weight* of v_T, denoted by

$$\alpha(v_T) = (a_1, \cdots, a_n) \in \mathbb{C}^n$$

to be the tuple of eigenvalues for $\rho(X_1), \cdots, \rho(X_n)$, respectively; that is $\rho(X_i)v_T = a_i v_T$ for all $i \in [n]$. Next, define

$$\mathrm{Spec}(n) := \{\alpha(v_T) : v_T \in \text{Young basis for some } \rho \in S_n^\wedge\} = \{\alpha(v_T) : T \text{ ends in } \rho,\ \rho \in S_n^\wedge\}.$$

We have already mentioned at the end of Section 2, that elements of the spectrum $\mathrm{Spec}(n)$ determine the Young vector v_T uniquely, up to a scalar. Therefore,

$$\dim \mathrm{GZ}(n) = |\,\mathrm{Spec}(n)| = \sum_{\rho \in S_n^\wedge} \dim \rho.$$

By definition of $\mathrm{Spec}(n)$, we see that there is a natural bijection between its elements and the paths in the branching graph of the chain $S_0 \subseteq S_1 \subseteq \cdots S_n$. Let $\alpha \to T_\alpha$ and $T \mapsto \alpha(v_T)$ be the bijection. Now we define the equivalence relation $\sim$ on $\mathrm{Spec}(n)$ as follows:

$$\alpha \sim \beta \iff T_\alpha \text{ and } T_\beta \text{ have the same end node}$$

or equivalently iff $v_\alpha := v_{T_\alpha}$ and $v_\beta := v_{T_\beta}$ belong to decomposition of the same irreducible representation of S_n. To quotient out $\mathrm{Spec}(n)$ by this relation means identifying all pahs with the same end. Hence, from the definition $\mathrm{Spec}(n)$, it is clear that $|\,\mathrm{Spec}(n)/\sim| = |S_n^\wedge|$. In the rest of this section, we describe this set and equivalence relation more precisely, and give an explicit description of the branching graph corresponding to the inductive chain of symmetric groups.

For any $i \in [n-1]$, define transpositions $s_i := (i\; i+1)$, called *Coxeter generators*. The proof of the following lemma can be found in [5].

Lemma 5 ([5], Section 4)**.** For any $\alpha = (a_1, \cdots, a_i, a_{i+1}, \cdots a_n) \in \mathrm{Spec}(n)$. Then for all i:
(1) $a_i \in \mathbb{Z}$;
(2) $a_i \neq a_{i+1}$;
(3) if $a_{i+1} = a_i \pm 1$, then $s_i \cdot v_\alpha = \pm v_\alpha$;
(4) if $a_{i+1} \neq a_i \pm 1$, then $\alpha' = s_i \cdot \alpha = (a_1 \cdots a_{i-1}\; a_{i+1}\; a_i \cdots a_n) \in \mathrm{Spec}(n)$ and $\alpha \sim \alpha'$.

We say that a Coxeter generator s_i is *admissible* for α if $a_{i+1} \neq a_i \pm 1$ for all i. We say that a vector $\alpha = (a_1, \cdots, a_n) \in \mathrm{Cont}(n)$ is a *content vector* if the following conditions are satisfied:
(1) $a_1 = 0$;
(2) $\{a_q - 1, a_q + 1\} \cap \{a_1, \cdots, a_{q-1}\} \neq \varnothing$;
(3) if $a_p = a_q = a$ for $p < q$, then $\{a-1, a+1\} \subseteq \{a_{p+1}, \cdots, a_{q-1}\}$.

Note that the first and the second condition force a_2 to be either 1 or -1. Similarly, $a_3 - 1$ and $a_3 + 1$ can take values $0, 1, -1$, so possible positive values of a_3 are 1 and 2 and possible negative values of a_3 are -1 and -2. In general, if $a_q > 0$, it has to be the case that $a_i = a_q - 1$ for some $i < q$, and if $a_q < 0$, it has to be the case that $a_i = a_q + 1$ for some $i < q$. It is obvious from the second condition that each a_q is an integer.

Example 1. One can check that $(0, 1, -1, 0, -2, 2, 1) \in \mathrm{Spec}(7)$. The first and second conditions are clearly satisfied. To see that the third condition holds, first note that $a_1 = a_4 = 0$ and $\{-1, 1\} = \{a_2, a_3\}$, as desired. Similarly, $a_2 = a_7 = 1$ and $\{0, 2\} \subseteq \{-1, 0, -2, 2\} = \{a_3, a_4, a_5, a_6\}$, as desired.

Let $\mathbb{Y}$ denote the *Young graph*, the graph whose vertices are Young diagrams, and two vertices μ and ν are joined by an edge $\nu \to \mu$ iff μ/ν is one single box. Recall that the number of paths from $\varnothing$ to μ in the Young graph is in bijection with the number of standard young tableaux on μ. Let $\mathrm{Tab}(n)$ denote the set of all paths from $\varnothing$ to all μ where $\mu \vdash n$. For any Young diagram (drawn in the English notation), we may define its content as follows. Viewing a Young diagram μ

as a (partial) matrix, we may enumerate each cell by an ordered pair (i, j) where i and j are the row and the column indices, respectively. We then define the *content* of each box (i, j) in μ to be $c(i, j) = j - i$. Unsurprisingly, the content vectors defined before and contents of boxes in Young diagrams are related. In fact, one can easily check that the sets $\mathrm{Tab}(n)$ and set of all content vectors of length n, $\mathrm{Cont}(n)$, are in bijection. For any $T = \mu_0 \to \mu_1 \to \cdots \to \mu_n \in \mathrm{Tab}(n)$, this bijection is defined by sending $T \mapsto (c(\mu_1/\mu_0), \cdots, c(\mu_n/\mu_{n-1}))$. We demonstrate this bijection by an example.

Example 2. Let $\mu = (5, 4, 2, 1) \vdash 12$. In the below figure, in the tableaux on the right, the number inside each box is its content.

1	3	6	10	12
2	4	8	9	
5	7			
11				

0	1	2	3	4
-1	0	1	2	
-2	-1			
-3				

Given some standard Young tableaux (or, equivalently, a path from $\varnothing$ to μ) as on the left in the figure above, we obtain the content vector α as follows. For each $k \in [12]$, we look at the cell of the tableaux (i, j) that k is in, and set $a_k := c(i, j)$. So, we obtain $\alpha = (0, -1, 1, 0, -2, 2, -1, 1, 2, 3, -3, 4)$. It is also clear that we can go back by reversing the steps (taking into an account the rules for standard Young tableaux when two cells have the same content).

On the set $\mathrm{Cont}(n)$, we define the following equivalence relation $\approx$:

$$\alpha \approx \beta \iff \alpha \text{ can be obtained from } \beta \text{ by a sequence of admissible permutations.}$$

Let $\alpha, \beta \in \mathrm{Cont}(n)$ and let T and S be the respective paths in $\mathrm{Tab}(n)$, via the bijection described earlier. It can be shown (refer to [1]) that $\alpha \approx \beta$ iff T and S are standard Young tableaux of the same shape (iff the corresponding paths have the same end). Our goal in the rest of this paper is to show that $\mathrm{Spec}(n) = \mathrm{Cont}(n)$, that $\sim$ and $\approx$ coincide, and that the branching graph for symmetric groups is isomorphic to the Young graph.

Theorem 6 ([5], Section 5)**.** $\mathrm{Spec}(n) \subseteq \mathrm{Cont}(n)$.

Proof. Let $\alpha = (a_1, \cdots, a_n) \in \mathrm{Spec}(n)$. We will show that α satisfies all three axioms of $\mathrm{Spec}(n)$. Note that since $X_1 = 0$, we have $\rho(X_1) = 0$, so $a_1 = 0$ and the condition (1) is satisfied. We prove the conditions (2) and (3) by induction on n. For $n = 2$, we have $X_2 = (1\ 2)$, so $v = (\rho(12))^2 v = a_2^2 v$ and thus $a_2 = \pm 1$. Thus (2) is satisfied for $n = 2$ and (3) is satisfied trivially. Suppose (2) and (3) hold for $n - 1$. Toward a contradiction, assume that $\{a_n - 1, a_n + 1\} \cap \{a_1, \cdots, a_{n-1}\} = \varnothing$. Then $a_{n-1} \neq a_n \pm 1$, so $a_n \neq a_{n-1} \pm 1$. Therefore, $(n-1\ n)$ is admissible for α. Hence, by Proposition 5, $(a_1, \cdots, a_{n-1}, a_n, a_{n-1}) \in \mathrm{Spec}(n)$. But then of course $(a_1, \cdots, a_{n-2}, a_n) \in \mathrm{Spec}(n-1)$, but neither $a_n \pm 1$ is in the set $\{a_1, \cdots, a_{n-2}\}$, contradicting the inductive hypothesis. So α satisfies (2). Next assume that $a_p = a_n = a$ for some $p \in [n]$ (note that for $q \neq n$, the claim holds by the inductive hypothesis). Toward a contradiction suppose $a_n - 1 \notin \{a_{p+1}, \cdots, a_{n-1}\}$. We may assume p is the largest possible index, i.e. $a \notin \{a_{p+1}, \cdots, a_{n-1}\}$. If $a+1$ occurred in the set $\{a_{p+1}, \cdots, a_{n-1}\}$ more than once, then by the inductive hypothesis, we would have that $a = (a + 1) - 1$ also occurs in this set more than once, which is impossible, as we assumed otherwise. Hence, $a+1$ occurs in that set at most once. If it doesn't occur at all, then by the series of admissible transpositions, we may permute the entries of α to obtain $\alpha' = (\cdots a\ a \cdots) \in \operatorname{Spec}(n)$. But this is a contradiction to part (2) of Lemma 5. If $a+1$ occurs in the set $\{a_{p+1}, \cdots, a_{n-1}\}$, then again by admissible transpositions we may transform α into $\alpha' = (\cdots, a\ a+1\ a \cdots)$. However, one can check that such combination is a contradiction to part (3) of Lemma 5 as well as the Coxeter relation $s_i s_{i+1} s_i = (i\ i{+}2) = s_{i+1} s_i s_{i+1}$. Hence, the condition (3) is satisfied as well, and we conclude that $\alpha \in \operatorname{Cont}(n)$.

$\square$

Remark 1. Note that if for some $\alpha \in \operatorname{Spec}(n)$ and $\beta \in \operatorname{Cont}(n)$ we have $\alpha \approx \beta$, then by definition of $\approx$ and part (4) of Lemma 5, we obtain $\beta \in \operatorname{Spec}(n)$ and $\alpha \sim \beta$. That is, $\approx$ is finer than $\sim$.

Theorem 7 ([1], Section 3.3)**.** The equality $\operatorname{Spec}(n) = \operatorname{Cont}(n)$ holds and the equivalence relations $\sim$ and $\approx$ coincide. The branching graph of the multiplicity-free inductive chain

$$S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n \subseteq S_{n+1} \subseteq \cdots$$

is isomorphic to the Young graph $\mathbb{Y}$.

Proof. Since $\operatorname{Cont}(n)$ and $\operatorname{Tab}(n)$ are in bijection, we know $|\operatorname{Cont}(n)| = |\operatorname{Tab}(n)|$. Since the relation $\approx$ identifies all standard Young tableaux of the same shape, we have $|\operatorname{Cont}(n)/\approx|$ is equal to the number of Young diagrams of size n, which is the number of partitions of n, denoted by $p(n)$. We have also seen that $|\operatorname{Spec}(n)/\sim| = |S_n^\wedge|$. Since the number of irreducible representations is equal to the number of conjugacy classes of S_n, and the latter is equal to the number of partitions of n, we have

$$|\operatorname{Cont}(n)/\approx| = p(n) = |\operatorname{Spec}(n)/\sim|.$$

Next, by Remark 1, we know that $\approx$ is finer than $\sim$, so every equivalence class of $\approx$ is a subset of an equivalence class of $\sim$. Thus, we have $|\operatorname{Spec}(n)/\sim| \leq |\operatorname{Spec}(n)/\approx|$. Since also $\operatorname{Spec}(n) \subseteq \operatorname{Cont}(n)$ by Theorem 6, we have $|\operatorname{Spec}(n)/\approx| \leq |\operatorname{Cont}(n)/\approx|$. Putting it all together, we have:

$$|\operatorname{Spec}(n)/\sim| \leq |\operatorname{Spec}(n)/\approx| \leq |\operatorname{Cont}(n)/\approx| = |\operatorname{Spec}(n)/\sim|.$$

Hence, all the weak inequalities above are in fact equalities. In particular, $|\operatorname{Spec}(n)/\approx| = |\operatorname{Cont}(n)/\approx|$ implies $\operatorname{Spec}(n) = \operatorname{Cont}(n)$ and $|\operatorname{Spec}(n)/\sim| = |\operatorname{Spec}(n)/\approx|$ implies that the two relations $\sim$ and $\approx$ coincide, as desired.

Finally, we show that the graphs are the same. Recall that the elements of $\operatorname{Spec}(n)$ are in bijection with the paths of the branching graph of the finite chain $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n$. The elements of $\operatorname{Cont}(n)$ are, in turn, in bijection with all the paths in the Young graph from $\varnothing$ to a partition of n. Thus, the equality $\operatorname{Spec}(n) = \operatorname{Cont}(n)$ gives us a natural bijection between the paths in the branching graph and the Young graph. Further, recall $\alpha \approx \beta$ in $\operatorname{Cont}(n)$ iff the two paths in the Young graph have the same end and $\sim$ is defined in the same way, but for the branching graph. Thus, the correspondence between the equivalence relations $\sim$ and $\approx$ implies that the vertices of the branching graph are in one-to-one correspondence with the vertices of $\mathbb{Y}$. In this case, correspondence of paths implies that the two graphs are isomorphic, as desired. $\square$

We have the following important corollary.

Corollary 2 ([5], Section 7)**.** The multiplicity of an irreducible representation π_μ of S_n in a representation π_λ of S_{n+k} is the number of paths between the diagrams λ and μ (where $\lambda \vdash n+k$, $\mu \vdash n$).

The above approach to the representation theory of S_n has many advantages, as many important well-known results follow naturally as corollaries. For example, the branching rule can be proven as a simple corollary of Corollary 2. For more exciting consequences, we refer the reader to [1].

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